INTEGRAL VALUES OF GENERATING FUNCTIONS OF RECURRENCE SEQUENCES

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Keywords: Generating function.

Abstract. Suppose that $a_0, a_1, \ldots$ is an integer sequence which satisfies a recurrence relation with constant coefficients, and let $T(x) = f(x)/g(x)$ be its generating function, where $f(x)$ and $g(x)$ have no common factors in $\mathbb{Z}[x]$. In this article, we study the problem of finding the rational values of x such that $T(x)$ is an integer. We say that such a number is *good* for the sequence. Our first main result is that if $g(x)$ has at least two different irreducible factors, or if $g(x)$ has a single irreducible factor of degree at least 3, then the sequence has only finitely many good values. We also study sequences of the form $0, 1, \ldots$ for which the recurrence relation has order 2. Among other results, we show that under a mild condition on the recurrence relation, the sequence has infinitely many good values, and we give a constructive method to find all of them.

1. Introduction

In this article, we study the generating functions of sequences defined by a recurrence relation with constant coefficients, and in particular seek rational inputs which result in integer outputs. We first encountered this problem in [3], in which Hong studied the generating function $F(x) = x/(1 - x - x^2)$ of the Fibonacci sequence

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2}, \quad (n \in \mathbb{Z}).$$

Hong showed that the values $x = F_n/F_{n+1}$ (with $n \geq 1$) always produce integer values of $F(x)$. He also proved a similar result for the generating function of the Lucas sequence. Shortly afterwards, Pongsriiam [8] solved the problem of finding all rational values of x which produce integer outputs for either of these sequences. Recently, the first named author [5] extended this work to any integer sequence $\{a_n\}_{n\in\mathbb{Z}}$ which satisfies the Fibonacci recurrence. In that article, it was proved, among other things, that if $F(x)$ is the generating function for an integer sequence satisfying the Fibonacci recurrence, then

- the numbers $x = F_n/F_{n+1}$ with $n \neq -1$ (including negative values of n) all produce integer values of $F(x)$, and
- the set of all rational x such that $F(x)$ is an integer can be divided into disjoint "families" in the sense that if p, q are integers with $\gcd(p, q) = 1$, then $F(p/q)$ is an integer if and only if both $F(q/(p+q))$ and $F((q-p)/p)$ are integers (assuming that none of the denominators in question are zero).

The objective of the present article is to prove similar results for the generating functions of other recurrence sequences. Suppose that $\{a_n\}_{n\geq 0}$ is an integer sequence in which the numbers $a_0, a_1, \ldots, a_{k-1}$ are given, and the recurrence satisfies the relation $a_n = N_1 a_{n-1} + \cdots + N_k a_{n-k}$ for some fixed integers $N_1, \ldots, N_k$. Writing $\mathbf{N} = (N_1, \ldots, N_k)$ and $\mathbf{a} = (a_0, \ldots, a_{k-1})$, define $T(x) = T_{\mathbf{N},\mathbf{a}}(x)$ to be the generating function of this sequence. We say that a number x is *good* for the sequence, or alternatively that x is good for $T(x)$, if $x \in \mathbb{Q}$ and $T(x) \in \mathbb{Z}$. Our first main goal is to prove that under a very general condition, a sequence has only finitely many good values. Our first theorem is the following.

Theorem 1.1. *Let $f(x)$ and $g(x)$ be polynomials with integer coefficients such that $T_{\mathbf{N},\mathbf{a}}(x) = \frac{f(x)}{g(x)}$, where $\gcd(f(x), g(x)) = 1$. If $g(x)$ has an irreducible factor of degree at least 3 or if it has at least two different nonconstant irreducible factors, then $T_{\mathbf{N},\mathbf{a}}(x)$ has only finitely many good values.*

This immediately leads to the following corollary.

Corollary 1.2. *Suppose that $\{a_n\}$ is a sequence that satisfies the recurrence $a_n = N_1 a_{n-1} + \cdots + N_k a_{n-k}$. If $k \geq 3$ and the polynomial $1 - N_1 x - \cdots - N_k x^k$ is irreducible over $\mathbb{Z}$, then the sequence $\{a_n\}$ has only finitely many good values.*

We can also obtain the following corollary for second-order recurrences.

Corollary 1.3. *Suppose that $\{a_n\}$ is a sequence that satisfies the recurrence $a_n = N_1 a_{n-1} + N_2 a_{n-2}$, and suppose that the polynomial $1 - N_1 x - N_2 x^2$ factors as $(1 - t_1 x)(1 - t_2 x)$ over $\mathbb{Z}$. If $t_1 \neq t_2$ and the polynomial $a_0 + (a_1 - N_1 a_0)x$ is not divisible by either $1 - t_1 x$ or $1 - t_2 x$, then the sequence $\{a_n\}$ has only finitely many good values. If $t_1 = t_2$ and $a_0 + (a_1 - N_1 a_0)x$ is not divisible by $1 - t_1 x$, then the sequence has infinitely many good values. Regardless of whether or not $t_1 = t_2$, if the polynomial*

$a_0+(a_1-N_1a_0)x$ can be written as $a_0(1-t_2x)$, then the sequence has infinitely many good values which are exactly the numbers $x=(k-a_0)/t_1k$, where $k\in\mathbb{Z}$ and $k\neq 0$.

Our second objective in this paper is to study more carefully the situation in which the recurrence relation has order 2, i.e., when $k=2$. In this situation, our goal is not merely to determine whether the number of good values is finite or infinite, but to also obtain some information about the structure of the good values for a sequence. To do this, we confine our attention to sequences with $a_0=0$ and $a_1=1$. For convenience, we slightly change our notation and write the recurrence relation as $a_n=Ma_{n-1}+Na_{n-2}$, calling the coefficients M and N instead of N_1 and N_2. In this context, we prove several theorems.

Theorem 1.4. *Let $\{a_n\}$ be the sequence with $a_0=0$ and $a_1=1$ which satisfies the recurrence $a_n=Ma_{n-1}+Na_{n-2}$ for $n\geq 2$. Suppose that $N>0$ and that $4N+M^2$ is a perfect square. Then the only good value for $T(x)$ is $x=0$.*

We note that this theorem cannot in general be extended to the case where $N<0$. In this setting, we have $T(x)=x/(1-Mx-Nx^2)$. If $4N+M^2$ is a square, then the denominator of $T(x)$ is reducible. If $4N+M^2>0$, then the two irreducible factors are different, and hence Corollary 1.3 implies that there are only finitely many good numbers. However, if $N<0$, then it could happen that there are more good numbers than just $x=0$. Although the set of good numbers can be found through the method given in the proof of Theorem 1.1, we do not have a simple formula that gives their values exactly. When $4N+M^2=0$, the denominator can be written as $(1-mx)^2$, where $m=M/2$ is an integer. In this case, it turns out that we are able to give an exact formula for all the good numbers, and we do so in the next theorem.

Theorem 1.5. *Let $\{a_n\}$ be the sequence with $a_0=0$ and $a_1=1$ which satisfies the recurrence $a_n=Ma_{n-1}+Na_{n-2}$ for $n\geq 2$, where $M,N\neq 0$, and suppose that $4N+M^2=0$. Then the sequence has infinitely many good values, and all the good values can be found as follows. Write $M/2=d_1d_2$, where $d_1>0$ and $\gcd(d_1,d_2)=1$, and let u_0,p_0 be integers with $d_1u_0-d_2p_0=1$. Then if $t\in\mathbb{Z}$ and $u_0+td_2\neq 0$, the number*

$$x=\frac{p_0+td_1}{d_1^2(u_0+td_2)}$$

is good for the sequence. All of the good numbers for the sequence can be found in this manner, when all different factorizations of $M/2$ as above are considered.

Although we cannot extend Theorem 1.4 to negative values of N, we are able to get a result in the special case when $M = 1$. We do this in the following theorem.

Theorem 1.6. *Let $\{a_n\}$ be a sequence with $a_0 = 0$ and $a_1 = 1$ which satisfies a recurrence of the form $a_n = a_{n-1} + Na_{n-2}$ for $n \geq 2$. If $N < -1$, then $x = 0$ is the only number that is good for $T(x)$, and if $N = -1$, then $x = 0$ and $x = 1$ are the only good numbers for $T(x)$.*

Finally, we complete our second objective by proving Theorem 1.7 below. In this theorem, we show that under some mild conditions, sequences coming from second-order recurrence relations have infinitely many good values, which can naturally be divided into families, and also show how to compute all of the good values. By the phrase "almost disjoint" below, we mean that the good value $x = 0$ may appear in more than one family, but that otherwise the families are disjoint.

Theorem 1.7. *Let $\{a_n\}$ be a sequence with $a_0 = 0$ and $a_1 = 1$ which satisfies a recurrence of the form $a_n = Ma_{n-1} + Na_{n-2}$ for $n \geq 2$. Suppose that $N > 0$, that $\gcd(M, N) = 1$ and that $4N + M^2$ is not a perfect square. Then there are infinitely many good values for $T(x)$. These values can be divided into finitely many almost disjoint families $p_0/q_0, p_1/q_1, \ldots$ in such a manner that there is a number x_1 such that*

$$p_n = 2x_1 p_{n-1} - p_{n-2} \quad \text{and} \quad q_n = 2x_1 q_{n-1} - q_{n-2} \quad \text{for } n \geq 2.$$

The ideas behind the proof of Theorem 1.1 are not too difficult. Suppose that $T(x)$ is the generating function for the sequence and that p and q are relatively prime integers such that $x = p/q$ is good for the sequence. Considering $T(p/q)$ as a function of p and q, we may rewrite this function so that $T(p/q) = f_1(p,q)/g_1(p,q)$, where $f_1(p,q)$ and $g_1(p,q)$ are homogeneous polynomials, where any common factors have been cancelled. We then show that in order for $x = p/q$ to be good, $g_1(p,q)$ must be one of finitely many values. If $g_1(p,q)$ has at least two different irreducible factors, then it follows from Bezout's theorem that each of these values can be obtained from only finitely many choices of (p,q). Otherwise, $g_1(p,q)$ has only one irreducible factor (perhaps with multiplicity), whose degree is at least 3, and the conclusion follows from Thue's theorem.

In the proof of Theorem 1.7, the equivalence between good values and solutions of the equation $d_1u^2 - Muv - d_2v^2 = 1$ comes from simple algebraic considerations. Once we have this equivalence, we see that finding integer values of (u, v) depends on knowing which values of v make the expression $(4N + M^2)v^2 + 4d_1$ a perfect square. We are able to deal with this question through the theory of Pell equations. This theory also gives us the recurrence relations for p_n and q_n.

The structure of this article is as follows. In Section 2, we find a formula for $T(x)$ and $T(p/q)$, and state some of the results we will need in the proofs. We prove Theorem 1.1 and its corollaries in Section 3, and give some examples of these results in Section 4. Then in Section 5, we prove our results for second-order recurrences. Finally, we give a few more examples in Section 6.

2. Preliminaries

Suppose, as in the introduction, that $\{a_n\}_{n\geq 0}$ is a sequence of integers in which $a_0, \dots, a_{k-1}$ (not all equal to 0) are given, and such that $a_n = N_1a_{n-1} + N_2a_{n-2} + \cdots + N_ka_{n-k}$ for $n \geq k$, where the integers $N_1, \dots, N_k$, with $N_k \neq 0$, are given. We begin by finding a formula for $T(x) = T_{\mathbf{N},\mathbf{a}}(x)$, the generating function for this sequence.

Lemma 2.1. *The generating function for the sequence $a_0, a_1, \dots$ is*

$$T(x) = T_{\mathbf{N},\mathbf{a}}(x) = \frac{a_0 + \sum_{j=1}^{k-1}\left(a_j - \sum_{i=1}^{j} a_{j-i}N_i\right)x^j}{1 - \sum_{i=1}^{k} N_ix^i}. \tag{1}$$

Proof. The generating function of the sequence is $T(x) = \sum_{n\geq 0} a_n x^n$. Thus,

$$\begin{aligned}
T(x) &= \sum_{j=0}^{k-1} a_j x^j + \sum_{n\geq k} a_n x^n \\
&= \sum_{j=0}^{k-1} a_j x^j + \sum_{n\geq k} \left(\sum_{i=1}^{k} N_i a_{n-i} \right) x^n \\
&= \sum_{j=0}^{k-1} a_j x^j + \sum_{i=1}^{k} \left(\sum_{n\geq k} a_{n-i} x^{n-i} \right) N_i x^i \\
&= \sum_{j=0}^{k-1} a_j x^j + \sum_{i=1}^{k} \left(T(x) - \sum_{n=0}^{k-i-1} a_n x^n \right) N_i x^i \\
&= \sum_{j=0}^{k-1} a_j x^j + \left(\sum_{i=1}^{k} N_i x^i \right) T(x) - \sum_{i=1}^{k} \left(\sum_{n=0}^{k-i-1} a_n x^n \right) N_i x^i \\
&= \sum_{j=0}^{k-1} a_j x^j - \sum_{j=1}^{k-1} \left(\sum_{i=1}^{j} a_{j-i} N_i \right) x^j + \left(\sum_{i=1}^{k} N_i x^i \right) T(x) \\
&= a_0 + \sum_{j=1}^{k-1} \left(a_j - \sum_{i=1}^{j} a_{j-i} N_i \right) x^j + \left(\sum_{i=1}^{k} N_i x^i \right) T(x).
\end{aligned}$$

This immediately leads to the formula for $T(x)$ in the statement of the lemma. □

In Section 3, we will need the following theorems about solutions of equations and systems of equations. Note that Lemma 2.2 is a weak form of Bezout's Theorem.

Lemma 2.2 (Bezout's Theorem [2, Exercise 13.17])**.** *If F and G are polynomials in two variables over a field K, having degrees d and e respectively, and without common factors, then their intersection consists of at most $d \cdot e$ points.*

Lemma 2.3 (Thue's Theorem [9])**.** *Let $f(x, y)$ be an irreducible binary form with integer coefficients and degree $n \geq 3$. Then the equation $f(x, y) = m$, where m is an integer, has only finitely many solutions in integers x and y.*

In our first example in Section 4, we will make use of the following theorem, which was proved independently by Delone [1] and Nagell [7]. We note that the theorems in [1] and [7] contain somewhat more information than this.

Lemma 2.4. *If $\Delta(p,q)$ is an irreducible binary cubic form with integer coefficients and negative discriminant, then the equation $\Delta(p,q)=1$ has at most five solutions in integers. Moreover, if the equation has five solutions, then the discriminant is -23, and if there are four solutions, then the discriminant is either -31 or -44.*

In the proof of Theorem 1.7, we will need to make use of the following result, which comes from the theory of continued fractions. This lemma is Proposition 4.32 on page 176 of [6].

Lemma 2.5. *Suppose that A is a positive integer which is not a perfect square, and let $\alpha = x_1 + y_1\sqrt{A} > 1$, where (x_1, y_1) is the fundamental solution of the equation $x^2 - Ay^2 = 1$. Given a positive integer c for which there are nonnegative integers v and w satisfying $w^2 - Av^2 = c$, then there exist nonnegative integers n, v_0 and w_0 such that $\sqrt{c} \le w_0 + v_0\sqrt{A} < \alpha\sqrt{c}$, and $w_0^2 - Av_0^2 = c$, and $w + v\sqrt{A} = (w_0 + v_0\sqrt{A})\alpha^n$.*

In [6], it is only stated that, $w_0 + v_0\sqrt{A} < \alpha\sqrt{c}$, but this inequality can be replaced by $\sqrt{c} \le w_0 + v_0\sqrt{A} < \alpha\sqrt{c}$.

3. A Condition Guaranteeing Finitely Many Good Numbers

We begin this section by giving a proof of Theorem 1.1, which we repeat here for convenience.

Theorem 1.1. *Let $f(x)$ and $g(x)$ be polynomials with integer coefficients such that $T_{\mathbf{N},\mathbf{a}}(x) = \frac{f(x)}{g(x)}$, where $\gcd(f(x), g(x)) = 1$. If $g(x)$ has an irreducible factor of degree at least 3 or if it has at least two different nonconstant irreducible factors, then $T_{\mathbf{N},\mathbf{a}}(x)$ has only finitely many good values.*

Proof. From Lemma 2.1, we have $\deg f(x) < \deg g(x) \le k$. (We could have $\deg g(x) < k$ if common factors have been cancelled from (1) to make the numerator and denominator relatively prime.) Write $\deg g(x) = k_1$ and $\deg f(x) = k_2$, and for $p, q \in \mathbb{Z}$ with $q \ne 0$ and $\gcd(p,q) = 1$ define $g_1(p,q) = q^{k_1} g(p/q)$ and $f_1(p,q) = q^{k_2} f(p/q)$. Observe that f_1 and g_1 are homogeneous polynomials in two variables with integer coefficients, and that we have $T_{\mathbf{N},\mathbf{a}}(p/q) = q^{k_1-k_2} f_1(p,q)/g_1(p,q)$.

Since $f(x)$ and $g(x)$ are coprime over $\mathbb{Z}[x]$, they are also coprime over $\mathbb{Q}[x]$. Thus, there exist $\tilde{u}(x), \tilde{v}(x) \in \mathbb{Q}[x]$ such that

$$\tilde{u}(x)f(x) + \tilde{v}(x)g(x) = 1, \ \deg \tilde{u}(x) < k_1, \ \text{and} \ \deg \tilde{v}(x) < k_2.$$

Let A be the smallest positive integer such that $u(x) = A\tilde{u}(x)$ and $v(x) = A\tilde{v}(x)$ are polynomials with integer coefficients. Then, the relation

$$u(x)f(x) + v(x)g(x) = A, \ \deg u(x) < k_1 \text{ and } \deg v(x) < k_2 \tag{2}$$

holds, where all the polynomials have integer coefficients. We also have $\deg u(x) + k_2 = \deg v(x) + k_1$, since the leading coefficient of the left side of (2) must vanish. Let $u_1(p,q)$ and $v_1(p,q)$ be the homogeneous polynomials related to $u(x)$ and $v(x)$, respectively, so that $u_1(p,q) = q^{\deg u(x)}u(p/q)$ and $v_1(p,q) = q^{\deg v(x)}v(p/q)$. Then we have

$$\begin{aligned} A &= u(p/q)f(p/q) + v(p/q)g(p/q) \\ &= q^{-\deg u(x) - k_2}u_1(p,q)f_1(p,q) + q^{-\deg v(x)-k_1}v_1(p,q)g_1(p,q). \end{aligned} \tag{3}$$

Let p, q be relatively prime integers such that $x = p/q$ is a good number for $T_{\mathbf{N},\mathbf{a}}(x)$. Multiplying (3) by $q^{k_1+\deg u(x)}/g_1(p,q)$, we get

$$\frac{q^{k_1+\deg u(x)}A}{g_1(p,q)} = u_1(p,q)T_{\mathbf{N},\mathbf{a}}(p/q) + q^{k_1-k_2}v_1(p,q) \in \mathbb{Z}.$$

Observe that $g(x)$ is a divisor of $1 - \sum_{i=1}^{k} N_i x^i$. Thus we may suppose that

$$g(x) = 1 + \sum_{i=1}^{k_1} b_i x^i \quad \text{and} \quad g_1(p,q) = q^{k_1} + \sum_{i=1}^{k_1} b_i p^i q^{k_1-i},$$

with $b_{k_1} \neq 0$. Since $g_1(p,q)$ divides $q^{k_1+\deg u(x)}A$, we may write $g_1(p,q) = d_1 d_2$ with $d_1|A$ and $d_2|q^{k_1+\deg u(x)}$. Notice that there are only finitely many possible values for d_1 since A is fixed.

Next, let

$$e_2 = \gcd(g_1(p,q), q) = \gcd(b_{k_1}p^{k_1}, q) = \gcd(b_{k_1}, q).$$

Hence $e_2|b_{k_1}$, and therefore there are only finitely many possible values for e_2, since b_{k_1} is fixed. Also, every prime factor of d_2 must divide both $g_1(p,q)$ and q, and so those prime factors divide e_2. Therefore, we have $d_2|e_2^{k_1+\deg u(x)}$, and hence there are only finitely many possible values for d_2. Since both d_1 and d_2 have only finitely many possible values, there are only finitely many possible values for their product $d_1 d_2$.

Finally, write $g(x)$ as a product of irreducible factors

$$g(x) = h_1(x)^{\alpha_1} h_2(x)^{\alpha_2} \cdots h_s(x)^{\alpha_s}$$

over $\mathbb{Z}[x]$. Then

$$g_1(p,q) = \tilde{h}_1(p,q)^{\alpha_1} \tilde{h}_2(p,q)^{\alpha_2} \cdots \tilde{h}_s(p,q)^{\alpha_s},$$

where $\tilde{h}_i(p,q) = q^{\deg(h_i(x))} h_i(p/q)$, for $1 \le i \le s$. If $g_1(p,q) = d_1 d_2$, then there exist integers $c_1, c_2, \ldots, c_s$ such that

$$c_1^{\alpha_1} c_2^{\alpha_2} \cdots c_s^{\alpha_s} = d_1 d_2 \tag{4}$$

and $\tilde{h}_i(p,q) = c_i$ for $1 \le i \le s$. If $s \ge 2$, then Bezout's theorem (Lemma 2.2) implies that the system $\tilde{h}_i(p,q) = c_i$, $(1 \le i \le s)$, has only a finite number of solutions. Since there are only finitely many possibilities for the integers $c_1, c_2, \ldots, c_s$ which satisfy (4), the equation $g_1(p,q) = d_1 d_2$ has a finite number of solutions.

If $s = 1$ and $\deg h_1 \ge 3$, then (4) becomes $c_1^{\alpha_1} = d_1 d_2$. Since $\deg h_1 \ge 3$, Thue's theorem (Lemma 2.3) guarantees that there are only finitely many integer solutions for $h_1(p,q) = c_1$. This completes the proof of the theorem. □

Note that the proof of Corollary 1.2 is now trivial. Suppose that $k \ge 3$ and the polynomial $1 - N_1 x - \cdots - N_k x^k$ is irreducible. Then this polynomial is $g(x)$ in Theorem 1.1, and we have $\gcd(f(x), g(x)) = 1$ since $g(x)$ is irreducible and $\deg f(x) < \deg g(x)$. Then Theorem 1.1 says that the sequence has only finitely many good values.

Finally, we prove Corollary 1.3, which we again restate for convenience.

Corollary 1.3. *Suppose that $\{a_n\}$ is a sequence that satisfies the recurrence $a_n = N_1 a_{n-1} + N_2 a_{n-2}$, and suppose that the polynomial $1 - N_1 x - N_2 x^2$ factors as $(1 - t_1 x)(1 - t_2 x)$ over $\mathbb{Z}$. If $t_1 \ne t_2$ and the polynomial $a_0 + (a_1 - N_1 a_0)x$ is not divisible by either $1 - t_1 x$ or $1 - t_2 x$, then the sequence $\{a_n\}$ has only finitely many good values. If $t_1 = t_2$ and $a_0 + (a_1 - N_1 a_0)x$ is not divisible by $1 - t_1 x$, then the sequence has infinitely many good values. Regardless of whether or not $t_1 = t_2$, if the polynomial $a_0 + (a_1 - N_1 a_0)x$ can be written as $a_0(1 - t_2 x)$, then the sequence has infinitely many good values which are exactly the numbers $x = (k - a_0)/t_1 k$, where $k \in \mathbb{Z}$ and $k \ne 0$.*

Proof. First, from Lemma 2.1, we see that before any possible reducing of fractions, the generating function for the sequence $\{a_n\}$ is given by

$$T(x) = \frac{a_0 + (a_1 - N_1 a_0)x}{1 - N_1 x - N_2 x^2} = \frac{a_0 + (a_1 - N_1 a_0)x}{(1 - t_1 x)(1 - t_2 x)}.$$

If $t_1 \neq t_2$ and the numerator is not divisible by either $1 - t_1 x$ or $1 - t_2 x$, then $g(x) = (1 - t_1 x)(1 - t_2 x)$ has two distinct irreducible factors, and the result follows from Theorem 1.1.

If $t_1 = t_2$ and there is no cancellation, then we have $T(p/q) = (Aq + Bpq)/(q - t_1 p)^2$, where $A = a_0$ and $B = a_1 - N_1 a_0$ are integers. Regardless of the values of A and B, we may find good values by letting p be any integer and setting $q = 1 + t_1 p$. Since different values of p yield different values of p/q, the sequence has infinitely many good values.

For the last statement, if $a_0 + (a_1 - N_1 a_0)x = a_0(1 - t_2 x)$, then after cancelling, the generating function becomes $T(x) = a_0/(1 - t_1 x)$. To find the good values, we let k be any integer and solve the equation $T(x) = k$. For $k = 0$, there are obviously no solutions since $a_0 \neq 0$. If $k \neq 0$, then simple algebra (along with the fact that $t_1 \neq 0$) shows that the equation $T(x) = k$ has the unique solution $x = (k - a_0)/t_1 k$. □

4. Examples

Example 4.1 (The Tribonacci Sequence)**.** In this first example, we calculate all the good values for the Tribonacci sequence, which has $a_0 = a_1 = 0$ and $a_2 = 1$, and satisfies the recurrence $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. The generating function for this sequence is $T(x) = x^2/(1 - x - x^2 - x^3)$. Since the denominator is irreducible over $\mathbb{Z}$, Theorem 1.1 implies that there are only finitely many good values. To find them, suppose that $x = p/q$ is good. When we rewrite $T(x)$ in terms of p and q, we have

$$T(p/q) = \frac{p^2 q}{q^3 - pq^2 - p^2 q - p^3}.$$

The denominator of this function is $g_1(p, q)$, and we have

$$\gcd(g_1(p, q), p) = \gcd(g_1(p, q), q) = 1.$$

Hence $g_1(p, q)$ divides 1, and we must have $g_1(p, q) = \pm 1$. Since $g_1(-p, -q) = -g_1(p, q)$, we need only consider $g_1(p, q) = 1$. We can check that the denominator has discriminant -44, and so Lemma 2.4 implies that the equation $g_1(p, q) = 1$

has at most four solutions. One can check with a computer algebra program such as Sage or Maple that the pairs $(p, q) = (-1, 0)$, $(0, 1)$, $(1, 2)$, and $(-56, -103)$ all satisfy the equation, and these must be all the solutions. The first pair does not lead to a good value since the denominator would be zero, and so the three numbers $x = 0$, $x = 1/2$, and $x = 56/103$ are the only good values for the Tribonacci sequence.

Example 4.2. Let $\{a_n\}$ be the sequence defined by

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 3, \quad \text{and} \quad a_n = a_{n-1} + a_{n-2} + a_{n-3}.$$

That is, we use the Tribonacci recurrence, but with different initial values. The generating function for this sequence is

$$T(x) = \frac{1 + x^2}{1 - x - x^2 - x^3}.$$

Note that we have $\gcd(1 + x^2, 1 - x - x^2 - x^3) = 1$. Moreover, we have $(1 - x - x^2 - x^3)(-1) + (x^2 + 1)(x + 1) = 2$, and so $A = 2$. Since $g_1(p, q) = q^3 - pq^2 - p^2q - p^3$ and $\gcd(p, q) = 1$, we have

$$e_2 = \gcd(g_1(p, q), q) = \gcd(-p^3, q) = 1.$$

Since $d_1|A$, we have $d_1 \in \{-2, -1, 1, 2\}$, and since $d_2|e_2^{2\deg(g(x))-1}$, we have $d_2|1$, and so $d_2 \in \{-1, 1\}$. Therefore $d_1d_2 \in \{-2, -1, 1, 2\}$, and we need to solve $g_1(p, q) = d_1d_2$ for each of these values. As in the previous example, since the function $g_1(p, q)$ is odd, we only need to solve $g_1(p, q) = 1$ and $g_1(p, q) = 2$. We have already found that the solutions of $g_1(p, q) = 1$ are

$$(p, q) = (-1, 0), (0, 1), (1, 2), (-56, -103),$$

and we can use PARI or other software to find that the solutions of $g_1(p, q) = 2$ are

$$(p, q) = (1, 1), (1, -1).$$

This gives the values

$$x = -1,\ 0,\ \frac{1}{2},\ \frac{56}{103},\ 1$$

as potential good values. Note that our proof does not guarantee that these values are good, only that all the good values appear in this list. However, it turns out that all of these numbers give integers when inserted into $T(x)$. Therefore, all five of these values are good, and they are the only good values for this sequence.

Example 4.3. Let $\{a_n\}$ be the sequence defined by

$$a_0 = 13, \quad a_1 = 19, \quad a_2 = 47, \quad \text{and} \quad a_n = a_{n-1} + 2a_{n-2} + 2a_{n-3}.$$

The generating function for this sequence is

$$T(x) = \frac{13 + 6x + 2x^2}{1 - x - 2x^2 - 2x^3}.$$

Note that the denominator is irreducible and that $\gcd(13 + 6x + 2x^2, 1 - x - 2x^2 - 2x^3) = 1$. From the relation

$$(1 - x - 2x^2 - 2x^3)(-1) + (13 + 6x + 2x^2)(-x + 2) = 25,$$

we may take $A = 25$, and so $d_1 \in \{-25, -5, -1, 1, 5, 25\}$. Moreover, we have $e_2 = \gcd(2, q)$, and so $e_2 | 2$. Since $d_2 | e_2^{2\deg(g(x))-1}$, we have $d_2 | 32$, and so

$$d_2 \in \{-32, -16, -8, -4, -2, -1, 1, 2, 4, 8, 16, 32\}.$$

As in the previous example, we need only consider the positive values of d_1 and d_2. Using PARI to solve the equations $g_1(p, q) = d_1 d_2$, with $g_1(p, q) = q^3 - pq^2 - 2p^2q - 2p^3$, we get the values in the table below.

d_1	d_2	d_1d_2	Solutions of $g_1(p,q) = d_1d_2$
1	1	1	$(0,1)$
1	2	2	$(-1,-2)$, $(-1,0)$, $(-1,1)$
1	4	4	$(-1,-1)$
1	8	8	$(0,2)$
1	16	16	$(-2,-4)$, $(-2,0)$, $(-2,2)$, $(3,7)$
1	32	32	$(-2,-2)$, $(-1,3)$
5	1	5	No solutions
5	2	10	$(-1,2)$, $(1,3)$
5	4	20	No solutions
5	8	40	$(-3,1)$
5	16	80	$(-2,4)$, $(2,6)$
5	32	160	No solutions
25	1	25	$(-2,-1)$
25	2	50	$(7,16)$
25	4	100	No solutions
25	8	200	$(-4,-2)$
25	16	400	$(14,32)$
25	32	800	No solutions

With these solutions, we see that the only potential good values for this sequence are

$$x = -3,\ -1,\ -\frac{1}{2},\ -\frac{1}{3},\ 0,\ \frac{1}{3},\ \frac{3}{7},\ \frac{7}{16},\ \frac{1}{2},\ 1,\ 2.$$

As in the previous example, we know that all of the good values for the sequence appear in this list, but we need to determine whether each value on the list is actually good. If we insert each of these eleven possibilities into $T(x)$, we find that only $T(0) = 13$ and $T(1/2) = -66$ are integers. Therefore, $x = 0$ and $x = 1/2$ are the only good values for this sequence.

5. Second-Order Recurrences

In this section, we confine our attention to sequences in which $a_0 = 0$ and $a_1 = 1$, and that satisfy a second-order recurrence of the form

$$a_n = Ma_{n-1} + Na_{n-2}$$

for $n \geq 2$. We assume that $N \neq 0$, as having $N = 0$ would yield a recurrence relation of order 1 instead of order 2. As mentioned in the introduction, our goal in this section is not merely to determine whether the number of good values for the sequence is finite or infinite, but rather to determine the structure of the set of good values. Thus we will consider some generating functions in this section in which the denominator is reducible, even though we already know from Corollary 1.3 that there are only finitely many good values. From Lemma 2.1, the generating function for this sequence is given by

$$T(x) = \frac{x}{1 - Mx - Nx^2}.$$

If $x = p/q$ with $\gcd(p, q) = 1$, then we have

$$T(p/q) = \frac{pq}{q^2 - Mpq - Np^2}.$$

Both of these generating functions are already in lowest terms regardless of whether the denominator is reducible.

These observations are already enough to prove Theorem 1.4, which we restate for convenience before giving the proof.

Theorem 1.4. *Let $\{a_n\}$ be the sequence with $a_0 = 0$ and $a_1 = 1$ which satisfies the recurrence $a_n = Ma_{n-1} + Na_{n-2}$ for $n \geq 2$. Suppose that $N > 0$ and that $4N + M^2$ is a perfect square. Then the only good value for $T(x)$ is $x = 0$.*

Proof. It is trivial to see that $x = 0$ is good for $T(x)$. We know that $T(p/q) \in \mathbb{Z}$ if and only if we have

$$pq = (q^2 - Mpq - Np^2)t$$

for some integer $t \neq 0$. This is equivalent to having

$$tq^2 - (Mpt + p)q - Np^2t = 0.$$

If we consider this as a quadratic equation with q as the variable, we find that

$$q = \frac{Mpt + p \pm \sqrt{(Mpt + p)^2 + 4Np^2t^2}}{2t} = \frac{Mpt + p \pm p\sqrt{(4N + M^2)t^2 + 2Mt + 1}}{2t}.$$

Write Δ for the expression under the square root.

Choose B such that $B^2 = 4N + M^2$ and B has the same sign as t. Note that we have $|B| > |M|$ since $N > 0$. This implies that

$$-2Bt + 1 < 2Mt + 1 < 2Bt + 1.$$

Since $\Delta = (Bt)^2 + 2Mt + 1$, this implies that

$$(Bt - 1)^2 < \Delta < (Bt + 1)^2.$$

Since $(Bt)^2 \neq \Delta$ is the only square between $(Bt-1)^2$ and $(Bt+1)^2$, we see that $\sqrt{\Delta}$ cannot be an integer. Therefore, if $p \neq 0$ is an integer and $T(p/q) \in \mathbb{Z}$, then p/q is not rational. Hence $x = 0$ is the only number that is good for $T(x)$. ☐

We now give the proof of Theorem 1.5, which we again restate.

Theorem 1.5. *Let $\{a_n\}$ be the sequence with $a_0 = 0$ and $a_1 = 1$ which satisfies the recurrence $a_n = Ma_{n-1} + Na_{n-2}$ for $n \geq 2$, where $M, N \neq 0$, and suppose that $4N + M^2 = 0$. Then the sequence has infinitely many good values, and all the good values can be found as follows. Write $M/2 = d_1 d_2$, where $d_1 > 0$ and $\gcd(d_1, d_2) = 1$, and let u_0, p_0 be integers with $d_1 u_0 - d_2 p_0 = 1$. Then if $t \in \mathbb{Z}$ and $u_0 + td_2 \neq 0$, the number*

$$x = \frac{p_0 + td_1}{d_1^2(u_0 + td_2)}$$

is good for the sequence. All of the good numbers for the sequence can be found in this manner, when all different factorizations of $M/2$ as above are considered.

Proof. Write $m = M/2$, and note that the condition $4N + M^2 = 0$ guarantees that $m \in \mathbb{Z}$. Also using the relation $4N + M^2 = 0$, we see that

$$T(x) = \frac{x}{(1 - mx)^2} \quad \text{and} \quad T(p/q) = \frac{pq}{(q - mp)^2}.$$

Suppose that $x = p/q$ is good for the sequence, where $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$. Then we have $\gcd(p, q - mp) = \gcd(p, q) = 1$. Now consider the number $q - mp$. If $q - mp < 0$, then we note that negating both q and p leads to the same good fraction, and so we may assume that $q - mp > 0$. Since p/q is good, we see that $q - mp$ divides q, and hence that $q - mp = \gcd(q, q - mp) = \gcd(q, m)$. Write $d_1 = q - mp$, so that $d_1 > 0$ as noted above, and define d_2 so that $m = d_1 d_2$. This gives us $T(p/q) = pq/d_1^2$, and since $\gcd(p, d_1) = 1$, we see that d_1^2 divides q. Define u so that $q = d_1^2 u$. Then we have $d_1^2 u - pm = d_1$, and hence $d_1 u - pd_2 = 1$. Therefore we have $\gcd(d_1, d_2) = 1$. Moreover, by the theory of linear Diophantine equations, if (u_0, p_0) is a particular solution of the equation $d_1^2 u - pm = d_1$, then there is an integer t such that $u = u_0 + td_2$ and $p = p_0 + td_1$. Therefore, if $x = p/q$ is any good value, then we can write

$$x = \frac{p_0 + td_1}{d_1^2(u_0 + td_2)},$$

where $d_1 > 0$, $\gcd(d_1, d_2) = 1$, and $d_1 d_2 = m$, the numbers u_0 and p_0 are as described above, and $t \in \mathbb{Z}$. Conversely, it is not difficult to check that any fraction defined in this way is good for $T(x)$ as long as the formula does not give a denominator equal to zero, which happens exactly when $u_0 + td_2 = 0$. This completes the proof of the theorem.

□

We now introduce some material that we need in the proofs of Theorems 1.6 and 1.7. We have already seen that

$$T(p/q) = \frac{pq}{q^2 - Mpq - Np^2}.$$

Since $\gcd(p, q) = 1$, we have

$$\gcd(p, q^2 - Mpq - Np^2) = \gcd(p, q^2) = 1.$$

Therefore, we see that if $x = p/q$ is good, then

$$\frac{q}{q^2 - Mpq - Np^2} \in \mathbb{Z}.$$

Moreover, since $(q^2 - Mpq - Np^2)|q$, we also have

$$\left|q^2 - Mpq - Np^2\right| = \gcd(q, q^2 - Mpq - Np^2) = \gcd(q, Np^2) = \gcd(q, N).$$

Therefore, see that $q^2 - Mpq - Np^2 = d_1$ is an integer divisor of both q and N. Consequently, there exists an integer u such that $q = d_1 u$. Define $d_2 \in \mathbb{Z}$ such that $N = d_1 d_2$. If we set $v = p$, then we can rewrite the equation $q^2 - Mpq - Np^2 = d_1$ as

$$d_1 u^2 - Muv - d_2 v^2 = 1. \tag{5}$$

Let $\mathcal{F}_{d_1,d_2} = \mathcal{F}_{d_1,d_2,M,N,\mathbf{a}}$ be the curve defined by (5). We have seen that each good value for $T(x)$ yields an integer point on some $\mathcal{F}_{d_1,d_2}$, and it is easy to check that an integer point (u, v) on $\mathcal{F}_{d_1,d_2}$ yields a good value $x = p/q$ by setting $(p, q) = (v, d_1 u)$, since $T(p/q) = \frac{pq}{d_1} = uv$. Thus the problem of finding all good numbers for $T(x)$ is equivalent to the problem of finding all the integer points on $\mathcal{F}_{d_1,d_2}$ for all pairs (d_1, d_2) with $d_1 d_2 = N$. Observe that if (u, v) is a point on $\mathcal{F}_{d_1,d_2}$, then $(-u, -v)$ is also a point on $\mathcal{F}_{d_1,d_2}$, and both points yield the same good value. Thus we may look only for points with $v > 0$. Also, one can check that (u, v) is a point on $\mathcal{F}_{d_1,d_2}$ if and only if (v, u) is a point on $\mathcal{F}_{-d_2,-d_1}$. Hence when N is positive, finding the points on all curves with $d_1 > 0$ also gives us the points on the curves with $d_1 < 0$.

We now begin the proofs of Theorems 1.6, and 1.7. As with our other theorems, we will restate them for convenience.

Theorem 1.6. *Let $\{a_n\}$ be a sequence with $a_0 = 0$ and $a_1 = 1$ which satisfies a recurrence of the form $a_n = a_{n-1} + Na_{n-2}$ for $n \geq 2$. If $N < -1$, then $x = 0$ is the only number that is good for $T(x)$, and if $N = -1$, then $x = 0$ and $x = 1$ are the only good numbers for $T(x)$.*

Proof. Note that we are assuming in this theorem that $M = 1$. As in the proof of Theorem 1.4, it is easy to see that $x = 0$ is good for $T(x)$.

By the observations above, if $x \neq 0$ is a good value for $T(x)$, then there exists a factorization $N = d_1 d_2$ such that $\mathcal{F}_{d_1,d_2}$ contains integral points (u, v) with $v \neq 0$. If $N < -1$ and $d_1 > 0$, then $d_2 < 0$ and an integral point with $v \neq 0$ satisfies

$$1 = d_1 u^2 - uv - d_2 v^2 \geq 1 + u^2 + v^2 - uv \geq 1 + |uv| \geq 2,$$

a contradiction. On the other hand, if $N < -1$ and $d_1 < 0$, then (5) can be rewritten as

$$Au^2 + uv + Bv^2 = -1,$$

where $A, B \geq 1$. However, these values of A and B guarantee that the function $Au^2 + uv + Bv^2$ has a global minimum of 0 at $(u, v) = (0, 0)$, and so (5) has no solutions.

If $N = -1$, then we must have either $(d_1, d_2) = (1, -1)$ or $(d_1, d_2) = (-1, 1)$. In the first case, $(u, v) = (1, 1), (-1, -1)$ are the only solutions of (5), since

$$u^2 - uv + v^2 = (u - v)^2 + uv.$$

These both lead to the good value $x = 1$. In the second case the equation (5) can be rewritten as $u^2 + uv + v^2 = -1$, and the same reasoning as above shows that this equation has no solutions. □

Theorem 1.7. *Let $\{a_n\}$ be a sequence with $a_0 = 0$ and $a_1 = 1$ which satisfies a recurrence of the form $a_n = Ma_{n-1} + Na_{n-2}$ for $n \geq 2$. Suppose that $N > 0$, that*

$\gcd(M,N)=1$ *and that* $4N+M^2$ *is not a perfect square. Then there are infinitely many good values for* $T(x)$. *These values can be divided into finitely many almost disjoint families* $p_0/q_0, p_1/q_1, \dots$ *in such a manner that there is a number* x_1 *such that*

$$p_n = 2x_1p_{n-1} - p_{n-2} \quad \textit{and} \quad q_n = 2x_1q_{n-1} - q_{n-2} \quad \textit{for } n \geq 2.$$

Proof. Let us write $A = 4N + M^2$. The equivalence between good values for the sequence and pairs (u, v) satisfying an equation of the form (5) is explained in the text before the proof of Theorem 1.6. Let us now consider positive integers d_1 and d_2 with $d_1d_2 = N$.

First, we show that the sets of good values coming from different curves $\mathcal{F}_{d_1,d_2}$ are disjoint. When we define our families of good values below, the good values in each family will come from the same curve, although a single curve might contain more than one family. Suppose that $(u, v) \in \mathcal{F}_{d_1,d_2}$. This leads to a good value $x = p/q$ with $p = v$ and $q = d_1u$. Let $g = \gcd(v, d_1u)$. Then g divides the left-hand side of (5), and hence divides 1. So $g = 1$ and the fraction p/q is in lowest terms. Suppose that this same value of x also arises from another point (u^*, v^*) on another curve $\mathcal{F}_{e_1,e_2}$. Then we have $x = p^*/q^*$ in lowest terms, where $p^* = v^*$ and $q^* = e_1u^*$. Since both fractions are in lowest terms, we have either $(p^*, q^*) = (p, q)$ or $(p^*, q^*) = (-p, -q)$. Therefore, we have $q^2 - Mpq - Np^2 = (q^*)^2 - Mp^*q^* - N(p^*)^2$. However, by the discussion in this section preceding the statement of Theorem 1.6, we have $q^2 - Mpq - Np^2 = d_1$ and $(q^*)^2 - Mp^*q^* - N(p^*)^2 = e_1$. Hence we have $(d_1, d_2) = (e_1, e_2)$, and the two curves $\mathcal{F}_{d_1,d_2}$ and $\mathcal{F}_{e_1,e_2}$ are really the same.

Next, if we consider (5) as an equation in the variable u, then its discriminant is

$$\Delta = (-Mv)^2 - 4 \cdot d_1 \cdot (-d_2v^2 - 1) = Av^2 + 4d_1.$$

Thus,

$$u = \frac{Mv \pm \sqrt{Av^2 + 4d_1}}{2d_1},$$

and we need to determine when $Av^2 + 4d_1$ is a perfect square. If the equation $w^2 - Av^2 = 4d_1$ has a solution, we can find all the solutions using Lemma 2.5.

Observe that we get all the solutions (u, v) with $v > 0$ by considering $w > 0$. Then any solution has the form $w_n + v_n\sqrt{A} = (w_0 + v_0\sqrt{A})\alpha^n$, where $\alpha > 0$ is defined in

Lemma 2.5 and (w_0, v_0) is a solution satisfying $2\sqrt{d_1} \le w_0 + v_0\sqrt{A} < 2\alpha\sqrt{d_1}$. All of these solutions (w_0, v_0), and there may be more than one, can be obtained using methods described in the literature. One such reference is [4, Section 16.3].

Since w_n and v_n can be written as linear combinations of α^n and $\bar{\alpha}^n$, they satisfy a recurrence relation whose characteristic equation is

$$x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha} = x^2 - 2x_1x + 1 = 0.$$

(Recall that $x_1 = (\alpha + \bar{\alpha})/2$, and also the number y_1 appearing below, are defined in Lemma 2.5.)

Thus, we have

$$w_n = 2x_1w_{n-1} - w_{n-2} \quad \text{and} \quad v_n = 2x_1v_{n-1} - v_{n-2}.$$

Given (w_0, v_0), we have the relation

$$w_1 + v_1\sqrt{A} = (w_0 + v_0\sqrt{A})(x_1 + y_1\sqrt{A}) = (x_1w_0 + Ay_1v_0) + (y_1w_0 + x_1v_0)\sqrt{A},$$

and so we can find the point (w_1, v_1). Then we may use the above recurrence relations to find (w_n, v_n) for all n.

Let (w_n, v_n) be a solution in nonnegative integers of

$$w^2 - Av^2 = 4d_1. \tag{6}$$

If we define

$$u_n = \frac{Mv_n \pm w_n}{2d_1},$$

then (u_n, v_n) is a point on the curve $\mathcal{F}_{d_1,d_2}$. Note that if M is odd, then so is A, and we can see from the relation (6) that v_n and w_n (and hence Mv_n and w_n) have the same parity. Similarly, if M is even, then so is A, and from (6) we see that w_n is also even. Therefore Mv_n and w_n have the same parity in this case as well. This implies that for any n, the expressions $(Mv_n + w_n)/2$ and $(Mv_n - w_n)/2$ are both integers.

Suppose first that $d_1 = 1$, so that $d_2 = N$. Then for every $n \ge 0$, both $(u, v) = \big((Mv_n + w_n)/2, v_n\big)$ and $(u, v) = \big((Mv_n - w_n)/2, v_n\big)$ are integer points on $\mathcal{F}_{1,N}$. These both correspond to values $x = p_n/q_n$, with $p_n = v_n$, which are good for $T(x)$. Since there are infinitely many values for v_n, and since we have already seen that all fractions of the form v_n/d_1u_n that we obtain are in lowest terms, we see that there are infinitely many good values for $T(x)$. In addition to these good values, we also have a sequence of good values $x = p_n/q_n$ with $p_n = u_n$ and $q_n = -Nv_n$, which come from the curve $\mathcal{F}_{-N,-1}$. We consider each of these sequences to be a family of good values.

Now suppose that $d_1 > 1$. Observe that $4d_1 = w_n^2 - (4N + M^2)v_n^2$ implies that

$$d_1 = -\left(\frac{Mv_n - w_n}{2}\right)\left(\frac{Mv_n + w_n}{2}\right) - Nv_n^2. \tag{7}$$

Since d_1 divides N, we know that it also divides the product on the right-hand side of (7). We have seen above that both of the terms in this product are integers, and we wish to determine which, if either, of them are divisible by d_1. We do this in the following lemma.

Lemma 5.1. *Let $d_1 > 1$. If d_1 divides $(Mv_0 + w_0)/2$, then the pairs $(\frac{Mv_n+w_n}{2d_1}, v_n)$ are integer solutions of (5), and d_1 does not divide $(Mv_n - w_n)/2$ for any $n \geq 0$. Conversely, If d_1 divides $(Mv_0 - w_0)/2$, then the pairs $(\frac{Mv_n-w_n}{2d_1}, v_n)$ are integer solutions of (5), and d_1 does not divide $(Mv_n + w_n)/2$ for any $n \geq 0$. Finally, if d_1 does not divide either $(Mv_0+w_0)/2$ or $(Mv_0-w_0)/2$, then d_1 does not divide either $(Mv_n + w_n)/2$ or $(Mv_n - w_n)/2$ for any $n \geq 0$.*

Before proving Lemma 5.1, we use it to finish the proof of Theorem 1.7. If d_1 does not divide either $(Mv_0 + w_0)/2$ or $(Mv_0 - w_0)/2$, then the curve $\mathcal{F}_{d_1,d_2}$ does not yield any good values for $T(x)$, and therefore neither does the curve $\mathcal{F}_{-d_2,-d_1}$. If $d_1|((Mv_0 + w_0)/2)$, then each point $(u, v) = \big((Mv_n + w_n)/2, v_n\big)$ on $\mathcal{F}_{d_1,d_2}$ yields a good value $x = p_n/q_n$ with $p_n = v_n$, and there is also a good value $x = p_n/q_n$ coming from $\mathcal{F}_{-d_2,-d_1}$ with $p_n = u_n$. Similarly, if $d_1|((Mv_0 - w_0)/2)$, then each point $(u, v) = \big((Mv_n - w_n)/2, v_n\big)$ on $\mathcal{F}_{d_1,d_2}$ yields a good value $x = p_n/q_n$ with $p_n = v_n$, and there is also a good value $x = p_n/q_n$ coming from $\mathcal{F}_{-d_2,-d_1}$ with $p = u_n$. Again, we consider each of these sequences to be a family of good values. Note that since there are finitely many possible values for d_1, and since each value of d_1 admits only finitely many possibilities for the point (w_0, v_0), there are only finitely many families of good values.

From the expression $u_n = \frac{v_n \pm w_n}{2d_1}$, we see that the sequence u_n also satisfies the recurrence relation

$$u_n = 2x_1u_{n-1} - u_{n-2}, \quad \text{for } n \geq 2.$$

In the good values, $x = p/q$, since $p = v_n$, we see that the values of p in each family satisfy the recurrence relation given in the theorem. Similarly, since the q-values in each family are a constant multiple of the u_n, they satisfy the same recurrence relation as the u_n, which is the relation given in the statement of the theorem. Finally, the same reasoning shows that the p- and q-values in the family coming from $\mathcal{F}_{-d_2,-d_1}$ also satisfy this recurrence.

Finally, we prove that the families of good values are disjoint except for $x = 0$. Suppose that (w, v) and (w^*, v^*) both lead to the same good value. We have already seen that (w, v) and (w^*, v^*) must lead to points (u, v) and (u^*, v^*) on the same curve $\mathcal{F}_{d_1,d_2}$, and the common good value is

$$x = \frac{v}{d_1 u} = \frac{v^*}{d_1 u^*}.$$

Since we have seen that both fractions are in lowest terms, we must either have $(u^*, v^*) = (u, v)$ or $(u^*, v^*) = (-u, -v)$. In the second case, since our method only gives non-negative values for v and v^*, we must have $v = v^* = 0$, which leads to the good value $x = 0$.

In the first case, either $w^* = w$, in which case we are done, or else one of u, u^* comes from the formula $(Mv_n - w_n)/2d_1$ and the other comes from the formula $(Mv_n + w_n)/2d_1$. This implies that $w^* = -w$, which as before implies that $w = 0$. We claim that this is impossible. If $w = 0$, then we must have $-Av^2 = 4d_1$. However, $A = 4N + M^2$ and $\gcd(M, N) = 1$. Combined with the fact that $d_1|N$, we see that $\gcd(d_1, A) = 1$, and therefore that $d_1|v^2$. If we write $v^2 = d_1 t$, we immediately get $-At = 4$, and so $A|4$. However, since $N > 0$, we see that $A \geq 4$, and we also have $A \neq 4$ because of the assumption that A is not a perfect square. Thus having $w = 0$ is not possible, and therefore we cannot have $(u^*, v^*) = -(u, v)$. This completes the proof that the families are disjoint (except for $x = 0$), and hence completes the proof of the theorem. □

In summary, in order to find all the good values for $T(x)$, for every positive divisor d_1 of N, we must find all solutions to equation (5). From these solutions, we deduce that $T(v/(d_1 u)) = T(u/(-d_2 v)) = uv$, and any value of $x \in \mathbb{Q}$ such that $T(x) \in \mathbb{Z}$ arises from one of these solutions. If N is not too large, we can use Lemma 2.5 to find the initial solutions.

We finish this section by giving the proof of Lemma 5.1.

Proof of Lemma 5.1. Assume first that there exists $n \geq 0$ such that d_1 divides both $(Mv_n + w_n)/2$ and $(Mv_n - w_n)/2$. In this case, we have $d_1 \mid w_n$ and $d_1 \mid Mv_n$. However, since $d_1|N$ and $\gcd(M, N) = 1$, we know that $\gcd(d_1, M) = 1$, and hence $d_1|v_n$. However, this implies that d_1^2 divides both terms on the right-hand side of (7), and therefore that $d_1^2|d_1$. This implies that $d_1 = 1$, which is excluded in the statement of this lemma. This proves that $d_1 > 1$ cannot divide both factors at the same time. To complete the proof, we must show that if d_1 divides $(Mv_r \pm w_r)/2$ for some $r \geq 0$, then d_1 divides $(Mv_n \pm w_n)/2$ for all $n \geq 0$.

Suppose that d_1 divides $(Mv_r + w_r)/2$ for some $r \geq 0$. From

$$w_{r+1} + v_{r+1}\sqrt{A} = (w_r + v_r\sqrt{A})(x_1 + y_1\sqrt{A}),$$

we obtain

$$\begin{aligned} \frac{Mv_{r+1} + w_{r+1}}{2} &= \frac{1}{2}\Big((My_1w_r + Mx_1v_r) + (x_1w_r + Ay_1v_r)\Big) \\ &= y_1\left(\frac{Mw_r + Av_r}{2}\right) + x_1\left(\frac{Mv_r + w_r}{2}\right). \end{aligned}$$

Since

$$Mw_r + Av_r = Mw_r + (4N + M^2)v_r = M(Mv_r + w_r) + 4d_1d_2v_r,$$

we obtain

$$\begin{aligned} &\frac{1}{2}\Big((My_1w_r + Mx_1v_r) + (x_1w_r + Ay_1v_r)\Big) \\ &\qquad = y_1M\left(\frac{Mv_r + w_r}{2}\right) + 2y_1d_1d_2v_r + x_1\left(\frac{Mv_r + w_r}{2}\right). \end{aligned}$$

Because d_1 divides $(Mv_r + w_r)/2$, we can see that d_1 also divides $(Mv_{r+1} + w_{r+1})/2$. Since d_1 divides $(Mv_n + w_n)/2$ for two consecutive subscripts, the recurrence relation shows that d_1 divides $(Mv_n + w_n)/2$ for all $n \geq 0$. The proof that if d_1 divides $(Mv_r - w_r)/2$ for some $r \geq 0$, then d_1 divides $(Mv_n - w_n)/2$ for all $n \geq 0$ is similar. This completes the proof of the lemma. □

6. Examples

Example 6.1. For our first example, we consider the sequence with $M = 1$ and $N = 4$. The generating function for this sequence is $T(x) = x/(1 - x - 4x^2)$.

We have $A = 4N + M^2 = 17$. Using Sage, we can check that the fundamental solution of $x^2 - 17y^2 = 1$ is $(x_1, y_1) = (33, 8)$. Thus we know that the numbers v_n, w_n, u_n, p_n, q_n all satisfy a recurrence relation whose characteristic equation is $x^2 - 66x + 1 = 0$. From Lemma 2.5, we need to find solutions to the equation $w^2 - 17v^2 = 4d_1$, for $d_1 \in \{1, 2, 4\}$, with

$$w + v\sqrt{17} < 2(33 + 8\sqrt{17})\sqrt{d_1}.$$

In particular, we need to have

$$0 \leq v < 2(33 + 8\sqrt{17})\sqrt{\frac{d_1}{17}}$$

For $d_1 = 1$, we find with Sage that the only solution in this range is $(w_0, v_0) = (2, 0)$, and this leads to $(w_1, v_1) = (66, 16)$. Since $d_1 = 1$, we have two sequences for u_n, one coming from $u_n = (Mv_n + w_n)/2$ and the other coming from $u_n = (Mv_n - w_n)/2$. The first sequence starts with $u_0 = 1$ and $u_1 = 41$, and the second sequence starts with $u_0 = -1$ and $u_1 = -25$. In the first sequence, the point $(u_0, v_0) = (1, 0)$ corresponds to the good value $x = p_0/q_0 = 0/1$, and the point $(u_1, v_1) = (41, 16)$ corresponds to the good value $x = p_1/q_1 = 16/41$. Using the recurrence relations for p_n and q_n, we obtain a family of good values

$$\frac{0}{1}, \frac{16}{41}, \frac{1056}{2705}, \frac{69680}{178489}, \frac{4597824}{11777569}, \ldots.$$

In the second sequence, the point $(u_0, v_0) = (-1, 0)$ again corresponds to $x = p/q = 0/(-1) = 0$, and the point $(u_1, v_1) = (-25, 16)$ corresponds to $x = p/q = -16/25$. This time, the recurrence relations for p_n and q_n give the family of good values

$$\frac{0}{-1}, \frac{16}{-25}, \frac{1056}{-1649}, \frac{69680}{-108809}, \frac{4597824}{-7179745}, \ldots.$$

Both of these families come from the curve $\mathcal{F}_{1,4}$. These same values of u_n and v_n also give us two families of good values for the sequence coming from the curve $\mathcal{F}_{-4,-1}$, using the formulas $p_n = u_n$ and $q_n = -4v_n$. Both of these families start with $n = 1$, since the zeroth term has $v_0 = 0$ in the denominator. From $(u_0, v_0) = (1, 0)$, we obtain the good values

$$\frac{41}{-64}, \frac{2705}{-4224}, \frac{178489}{-278720}, \frac{11777569}{-18391296}, \frac{777141065}{-1213546816}, \ldots$$

and from $(u_0, v_0) = (-1, 0)$, we obtain the good values

$$\frac{25}{64}, \frac{1649}{4224}, \frac{108809}{278720}, \frac{7179745}{18391296}, \frac{473754361}{1213546816}, \dots.$$

For $d_1 = 2$, there are two possible solutions for (w_0, v_0), and each gives rise to a family of good values from $\mathcal{F}_{2,2}$ and another family from $\mathcal{F}_{-2,-2}$. From the solution $(w_0, v_0) = (5, 1)$, we obtain $(w_1, v_1) = (301, 73)$. We can see that the numbers $u_n = (v_n - w_n)/2d_1$ are integers, and that $u_0 = -1$ and $u_1 = -57$. Since $(p_n, q_n) = (v_n, d_1 u_n)$, we obtain $(p_0, q_0) = (1, -2)$ and $(p_1, q_1) = (73, -114)$. This leads to the family of good numbers

$$\frac{1}{-2}, \frac{73}{-114}, \frac{4817}{-7522}, \frac{317849}{496338}, \dots.$$

These values of u_0 and v_0 also lead to a family coming from the curve $\mathcal{F}_{-2,-2}$:

$$\frac{1}{2}, \frac{57}{146}, \frac{3761}{9634}, \frac{248169}{635698}, \frac{16375393}{41946434}, \dots.$$

The second family for $d_1 = 2$ begins with $(w_0, v_0) = (29, 7)$, and leads to the family of good numbers

$$\frac{7}{18}, \frac{463}{1186}, \frac{30551}{78258}, \frac{2015903}{5163842}, \dots.$$

Again, we also get a family coming from the curve $\mathcal{F}_{-2,-2}$:

$$\frac{9}{-14}, \frac{593}{-926}, \frac{39129}{-61102}, \frac{2581921}{-4031806}, \frac{170367657}{-266038094}, \dots.$$

Finally, for $d_1 = 4$, there are three possibilities for the initial solution (w_0, v_0). First, we may take $(w_0, v_0) = (4, 0)$. However, for this solution, we see that neither $(Mv_0 + w_0)/2d_1 = 1/2$ nor $(Mv_0 - w_0)/2d_1 = -1/2$ is an integer. Thus, this initial solution does not yield any good values for the sequence. The other two possibilities, $(w_0, v_0) = (13, 3)$ and $(w_0, v_0) = (21, 5)$ each give two families of good values, one coming from $\mathcal{F}_{4,1}$ and another coming from $\mathcal{F}_{-1,-4}$. Although we do not give the details here, these families can be found in the same way as we have found the previous families.

Example 6.2. In this last example, we consider the sequence with $M = 1$ and $N = 29$. In this case, we have $A = 4N + M^2 = 117 = 3^2 \cdot 13$. Examining the equation $x^2 - 117y^2 = 1$, we find that the fundamental solution is $\alpha = 649 + 60\sqrt{117} = 649 + 180\sqrt{13}$. Thus we have $(x_1, y_1) = (649, 60)$. Since $N = 29$ is prime, the only possibilities for d_1 are 1 and 29. For $d_1 = 29$, when we use Lemma 2.5 to find initial solutions (w_0, v_0), we find that none exist. So neither the curve $\mathcal{F}_{29,1}$ nor the curve $\mathcal{F}_{-1,-29}$ leads to good values for this sequence.

If $d_1 = 1$ however, we find that there are three possibilities for (w_0, v_0), each of which yields two families of good values from $\mathcal{F}_{1,29}$ and two from $\mathcal{F}_{-29,-1}$. For $(w_0, v_0) = (2, 0)$, we have $(w_1, v_1) = (1298, 120)$. Also, we find that we may have either $u_0 = -1$ or $u_0 = 1$. The value $u_0 = -1$ comes from the formula $u_0 = (Mv_0 - w_0)/2$. From this value, we obtain $(p_0, q_0) = (0, -1)$ and $(p_1, q_1) = (120, -589)$. Since $x_1 = 649$, we know that both p_n and q_n satisfy a recurrence with characteristic equation $x^2 - 1298x + 1 = 0$. This yields a family of good values

$$\frac{0}{-1}, \frac{120}{-589}, \frac{155760}{-764521}, \frac{202176360}{-992347669}, \ldots$$

coming from $\mathcal{F}_{1,29}$. The corresponding family coming from $\mathcal{F}_{-29,-1}$ is

$$\frac{589}{3480}, \frac{764521}{4517040}, \frac{992347669}{5863114440}, \ldots.$$

Similarly, the value $u_0 = 1$ is found using the formula $u_0 = (Mv_0 + w_0)/2$. From this value we obtain $(p_0, q_0) = (0, 1)$ and $(p_1, q_1) = (120, 709)$, leading to the family of good values

$$\frac{0}{1}, \frac{120}{709}, \frac{155760}{920281}, \frac{202176360}{1194524029}, \ldots$$

coming from $\mathcal{F}_{1,29}$ and the family

$$\frac{709}{-3480}, \frac{920281}{4517040}, \frac{1194524029}{5863114440}, \ldots$$

coming from $\mathcal{F}_{-29,-1}$.

The other two possibilities for (w_0, v_0) are $(11, 1)$ and $(119, 11)$. Each of these leads to two families of good values from $\mathcal{F}_{1,29}$ and two families from $\mathcal{F}_{-29,-1}$ in the same way as above, although we omit the details.

Acknowledgements

The second and third authors were partially supported by FAPEMIG grant RED-00133-21. The second author was also partially supported by FAPEMIG grant APQ-02546-21, and the third author was also partially supported by FAPEMIG grant APQ-00470-22.

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